Shiga-Watanabe’s time inversion property
for self-similar diffusion processes

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Summary: Assume \((X_t, P^0)\) is \(1/2\)-self-similar, rotation invariant diffusion on \(R^d, d \geq 2\), starting at 0 and assume \(\{0\}\) is a polar set. We will show, using the corresponding well-known result for the radial process, that Shiga-Watanabe's time inversion property holds for \((X_t, P^0)\). The generalization for an \(\alpha\)-self-similar, rotation invariant diffusion, \(\alpha > 0\), is also given.

Key words: time inversion, self-similar, diffusion, rotation invariant, skew product, radial process, spherical process

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0. Introduction. Theorem.

The following time inversion property is well known for Brownian motion in \(R^d, d \geq 1\), and Bessel diffusions on \([0, \infty)\), starting at 0 (see Shiga, Watanabe (1973) and Watanabe (1975)):

\[
(X_t) \text{ has the same finite dimensional distributions as } (tX_{1/t}) \text{ under } P^0, \text{ for all } t>0. 
\]

This was in Graversen, Vuolle-Apiala (2000) shown to be true also for symmetrized Bessel processes on \(R\), starting at 0, in the case of the index \(\nu \in (-1,0)\), that is, \((X_t)\) can both hit 0 and can be started at 0. Symmetrized Bessel processes form the class of one dimensional rotation invariant \(1/2\)-self similar diffusions (see the definition below). If the index \(\nu \leq -1\) then 0 is an exit boundary point, that is, \((X_t)\) can hit 0 but it cannot be started there. If \(\nu \geq 0\) then 0 is an entrance boundary point, that is, \((X_t)\) can be started there and it will never come back. Thus in this case \((X_t)\) in fact lives either on \([0, \infty)\) or on \((-\infty, 0]\) and \(\{0\}\) is a polar set. Obviously, (0.1) is valid. (0.1) has been generalized for \(\alpha\)-self-similar diffusions on \([0, \infty)\) and for symmetric \(\alpha\)-self-similar diffusions on \(R, \alpha>0\), in Graversen, Vuolle-Apiala (2000). The corresponding generalization of (0.1) is then

\[
(X_t) \text{ has the same finite dimensional distributions as } (t^{2\alpha}X_{1/t}) \text{ under } P^0, \text{ for all } t>0. 
\]

In this note we will show that (0.2) holds for all rotation invariant, \(d\)-dimensional, \(\alpha\)-self-similar diffusions (that is, strong Markov processes with continuous paths), \(d \geq 2\), \(\alpha>0\), for which \(\{0\}\) is a polar set. Our main tool is a skew product representation for rotation invariant diffusions starting at 0; see Itô, Mc Kean Jr., 1974, p. 274 - 276.

Let \((X_t, P^0)\) be a rotation invariant (RI) \(\alpha\)-self-similar (\(\alpha\)-ss) diffusion on \(R^d, d \geq 2\), \(\alpha>0\), such that \(\{0\}\) is a polar set. By \(\alpha\)-self-similarity we mean that

\[
(X_t) \text{ under } P^0 \text{ has the same finite dimensional distributions as } (a^{-\alpha}X_{at}) \text{ under } P^{a^{-\alpha}} \text{ for all } x \in R^d, a>0
\]

and by (RI) that
(0.4) \( (X_t) \) under \( P^x \) has the same finite dimensional distributions as \( (T^{-1}(X_t)) \) under \( P^{T(x)} \) for all \( T \in O(d) \).

Self-similar diffusions on \( \mathbb{R} \) or on \( [0, \infty) \) are defined similarly. Brownian motion fulfills both (0.3) and (0.4). See more about processes fulfilling (0.3) and (0.4) in Graversen, Vuolle-Apiala (1986), Lamperti (1972) and Vuolle-Apiala, Graversen (1986). According to Graversen, Vuolle-Apiala (1986) and Vuolle-Apiala (2002), when \( X_0 \neq 0 \) the diffusion processes which fulfill (0.3) and (0.4) can be represented as skew products

(0.5) \( [|X_t|, \theta_A] \),

where \( A_t = \lambda \int_0^t |X_s|^{-1/\alpha} \, ds \) for some \( \lambda > 0 \), the radial part \( (|X_t|) \) is an \( \alpha \)-ss diffusion on \( (0, \infty) \), and \( (\theta_t) \) is a spherical Brownian motion on \( S^{d-1} \) independent of \( (|X_t|) \).

Remark: As showed in Graversen, Vuolle-Apiala (1986), (0.5) is valid for all strong Markov processes with cadlag paths fulfilling (0.3) and (0.4). However, as showed by a counterexample by J. Bertoin, W. Werner (1996), the independence between \( (|X_t|) \) and \( (\theta_t) \) is not necessarily true if the paths are only right continuous. There is an error in the proof of Proposition 2.4, p.19-20 in Graversen, Vuolle-Apiala (1986). It was showed in Vuolle-Apiala (2002), Lemma 2.1, that \( (|X_t|) \) and \( (\theta_t) \) are independent in the case of continuous paths.

We want to prove the following:

**Theorem:** Let \( (X_t, P^0) \) be an (RI) \( \alpha \)-ss diffusion on \( \mathbb{R}^d \), \( \alpha > 0, \ d \geq 2 \), starting at 0, having \( \{0\} \) as a polar set. Then the time inversion property (0.2) is valid.

1. **The Proof of the Theorem**

   The proof will be based on

   **Proposition:** Let \( (r_t) \) be an \( \alpha \)-ss diffusion on \( [0, \infty), \ \alpha > 0 \), such that 0 is an entrance, non-exit boundary point. Then the skew product

   (0.6) \( [r_t, \theta_{\lambda \int_0^t r_s^{1/\alpha} \, ds}] \), \( t > 0, \ r_0 > 0 \), \( \theta \in S^{d-1} \),

   \( (\theta_t, Q^\theta) \) is a spherical Brownian motion on \( S^{d-1} \) independent of \( (r_t) \), such that \( Q^\theta(\theta_0 = \theta) = 1 \ \forall \theta \in S^{d-1} \), can be completed to be an \( \alpha \)-ss diffusion on \( \mathbb{R}^d \) by defining

   (0.7) \( [r_t, \nu_{\lambda \int_0^t r_s^{1/\alpha} \, ds}] \), \( t > 0, \) when \( r_0 = 0 \).

   where \( (\nu_t, Q) \) is an independent, spherical Brownian motion defined for \(-\infty < t < +\infty \) and the law of \( (\nu_0) \) is the uniform spherical distribution \( m(d\theta) \).
Remark 1: Because of a uniqueness result of RI measures on $S^{d-1}$ there is at most one way to complete (0.6) to be RI on the whole $R^d$.

Remark 2: It is obvious that $(\nu_t, Q)$ in fact is a stationary process and $\nu_t$ is uniformly distributed for all $t \in R$ (see Kuznetsov, 1973).

We have

\[(0.8) \quad Q\{\nu_t \in d\theta_1, ..., \nu_n \in d\theta_n\} = m(d\theta_1)Q^{\theta_1}(\theta_{t+1} \in d\theta_2) ... Q^{\theta_{n-1}}(\theta_{t+n-1} \in d\theta_n)\]

for $-\infty < t_1 < ... < t_n < +\infty$.

In order to prove Proposition we need

Lemma 1: $(\nu_{t_1}, ..., \nu_{t_n})$ under $Q$ has the same distribution as $(\nu_{t_1}, ..., \nu_{t_1})$.

Proof: Follows immediately from (0.8) and the fact that $(\theta_t, Q^\theta)$ has a symmetric density with respect to the uniform measure $m(d\theta)$ on $S^{d-1}$ (see Vuolle-Apiala, Graversen, 1986, Lemma 3, p.329).

In the proof of Proposition we will use the result of Itô-McKean (1974), p. 275, which says that the skew product (0.6) can be completed to be a diffusion (which obviously is RI) on the whole $R^d$ having the skew product (0.7) when $r_0 = 0$ iff $A_{0+} = \infty$ a.s $P^0$. Here we need

Lemma 2: Let $(r_t)$ be an $\alpha$-ss diffusion on $[0, \infty)$ such that 0 is an entrance, non-exit boundary point. Then

\[P^0\{\int_0^t r_s^{-1/\alpha} ds = \infty\} = 1 \quad \forall \epsilon > 0.\]

Proof of Lemma 2: (0.3) implies that

\[P^0\{\int_0^\epsilon r_s^{-1/\alpha} ds = \infty\} = P^0\{\int_0^\epsilon (a^{(-\alpha)\epsilon})^{-1/\alpha} ds = \infty\} = P^0\{\int_0^\epsilon r_s^{-1/\alpha} ds = \infty\} = P^0\{\int_0^\infty r_s^{-1/\alpha} ds = \infty\} = 1.\]

So it suffices to show that

\[P^0\{\int_0^\infty r_s^{-1/\alpha} ds = \infty\} = 1.\]

The Markov property gives

\[P^0\{\int_0^\infty r_s^{-1/\alpha} ds = \infty\} \geq P^0\{\int_0^\infty r_s^{-1/\alpha} ds = \infty\} = E^0\{ P^n\{\int_0^\infty r_s^{-1/\alpha} ds = \infty\} \}\quad \forall t > 0.\]

Because 0 is an entrance, non-exit boundary point, $r_t > 0$ a.s. ($P^0$). Now, according to Lamperti (1972),

\[P^0\{\int_0^\infty r_s^{-1/\alpha} ds = \infty\} = 1\quad \text{for all } r > 0\]
and thus

\[ E^0 \{ \mathbb{P}^\nu \{ \int_0^\infty r_s^{-1/\alpha} ds = \infty \} \} = 1 \]

which implies

\[ \mathbb{P}^0 \{ \int_0^\infty r_s^{-1/\alpha} ds = \infty \} = 1. \]

**Proof of Proposition:** It only remains to prove that the skew product

\[ ([r_t, \nu] \lambda_{\text{r}^{-1/\alpha} ds} ) \text{ when } r_0 = 0, \]

fullfills the \( \alpha \)-self-similarity condition (0.3) under \( \mathbb{P}^0 \). Let \( I_1, \ldots, I_n \) be Borel subsets of \([0, \infty)\) and \( J_1, \ldots, J_n \) Borel subsets of \( S^{d-1} \). We will show

\[ \mathbb{P}^0 \{ r_t \in I_1, \ldots, r_n \in I_n, \nu \lambda_{\text{r}^{-1/\alpha} ds} \in J_1, \ldots, \nu \lambda_{\text{r}^{-1/\alpha} ds} \in J_n \} = \]

\[ \mathbb{P}^0 \{ \alpha^{-\alpha} r_{a_1} \in I_1, \ldots, \alpha^{-\alpha} r_{a_n} \in I_n, \nu \lambda_{\text{r}^{-1/\alpha} ds} \in J_1, \ldots, \nu \lambda_{\text{r}^{-1/\alpha} ds} \in J_n \} \]

for all \( t > 0 \).

For simplicity, assume \( n = 2 \), the general case is analogous.

Now the right hand side of (*) for \( n = 2 \) is equal to

\[ \mathbb{P}^0 \{ r_{t_1} \in I_1, r_{t_2} \in I_2, \nu \lambda_{\text{r}^{-1/\alpha} ds} \in J_1, \nu \lambda_{\text{r}^{-1/\alpha} ds} \in J_2 \} = \]

\[ \mathbb{P}^0 \{ \alpha^{-\alpha} r_{a_1} \in I_1, \alpha^{-\alpha} r_{a_2} \in I_2, \nu \lambda_{\text{r}^{-1/\alpha} ds} \in J_1, \nu \lambda_{\text{r}^{-1/\alpha} ds} \in J_2 \} \]

because \( (r_t) \) fullfills (0.3) and because of independence between \( (r_t) \) and \( (\nu_t) \).

This is further equal to

\[ \mathbb{P}^0 \{ r_{t_1} \in I_1, r_{t_2} \in I_2, \nu \lambda_{\text{r}^{-1/\alpha} ds + \lambda_{\text{r}^{-1/\alpha} ds} + \lambda_{\text{r}^{-1/\alpha} ds} + \lambda_{\text{r}^{-1/\alpha} ds} } \in J_1, \nu \lambda_{\text{r}^{-1/\alpha} ds + \lambda_{\text{r}^{-1/\alpha} ds} + \lambda_{\text{r}^{-1/\alpha} ds} + \lambda_{\text{r}^{-1/\alpha} ds} } \in J_2 \} = \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{P}^0 \{ r_{t_1} \in I_1, r_{t_2} \in I_2, \lambda_{\text{r}^{-1/\alpha} ds} \in du, \lambda_{\text{r}^{-1/\alpha} ds} \in dv, \lambda_{\text{r}^{-1/\alpha} ds} \in dw, \lambda_{\text{r}^{-1/\alpha} ds} \in dw \} \]

\[ \nu_{u+v} \in J_1, \nu_{u+w} \in J_2 \} = \]
because of independence between \((r)\) and \((\nu)\). Now \((\nu)\) is a stationary process and thus this is equal to

\[
\int \int \int \mathbb{P} \{ r_1, r_2 \in I_1, I_2, u^{-1/2} \int_1^t r_1^{1/2} \mathrm{d}s \in \mathrm{d}u, v^{-1/2} \int_1^t r_2^{1/2} \mathrm{d}s \in \mathrm{d}v, w^{-1/2} \int_1^t r_3^{1/2} \mathrm{d}s \in \mathrm{d}w \} \prod(\nu_{u+v}, J_1, \nu_{u+w}, J_2) =
\]

\[
\int \int \int \mathbb{P} \{ r_1, r_2 \in I_1, I_2, u^{-1/2} \int_1^t r_1^{1/2} \mathrm{d}s \in \mathrm{d}u, v^{-1/2} \int_1^t r_2^{1/2} \mathrm{d}s \in \mathrm{d}v, w^{-1/2} \int_1^t r_3^{1/2} \mathrm{d}s \in \mathrm{d}w \} \prod(\nu_v, J_1, \nu_w, J_2) =
\]

\[
P^0 \{ r_1, r_2 \in I_1, I_2, u^{-1/2} \int_1^t r_1^{1/2} \mathrm{d}s \in \mathrm{d}u, v^{-1/2} \int_1^t r_2^{1/2} \mathrm{d}s \in \mathrm{d}v \} \prod(\nu_v, J_1, \nu_w, J_2) =
\]

Now we can prove Theorem:

**Proof of Theorem:** \((X_t)\) has according to Graversen, Vuolle-Apiala (1986), Vuolle-Apiala (2002) and Proposition a skew product representation

\[
[r_t, \theta_t, \lambda^{-1} \int_0^t r_s \ \mathrm{d}s] \quad \text{as } X_0 = 0
\]

and

\[
[r_t, \nu_t, \lambda^{-1} \int_0^t r_s \ \mathrm{d}s] \quad \text{as } X_0 = 0,
\]

where \((r)\) is the radial process, \((\theta_t, Q^\theta)\) is an independent spherical Brownian motion such that \(Q^\theta(\theta_0=\theta)=1\) and \((\nu_t, Q)\) is an independent, stationary, spherical Brownian motion defined for \(-\infty < h < +\infty\) and the law of \(\nu_h\) is the uniform spherical distribution for all \(h \in \mathbb{R}\). To show (0.2) let us consider the distribution of \(\{t_1^{1/2}X_{1t_1}, \ldots, t_2^{\alpha_1}X_{1t_2}\}\) under \(P^0\). Assume for simplicity \(n=2, \alpha=1/2\), the general case is analogous. Let \(I_i, J_i, i=1,2\), be Borel subsets of \((0, \infty)\) and \(S^{d-1}\), respectively. Now

\[
P^0 \{ t_1 X_{1t_1} \in (I_1, J_1), t_2 X_{1t_2} \in (I_2, J_2) \} =
\]

\[
P^0 \{ t_1 r_{1t_1} \in (I_1, J_1), t_2 r_{1t_2} \in (I_2, J_2) \} =
\]
\[
\int_\infty^{-\infty} \int_\infty^{-\infty} P^0 (t_1 r_{1/t_1} \in I_1, t_2 r_{1/t_2} \in I_2, \nu_u \in J_1, \nu_v \in J_2, \lambda \int_1^{1/t_1} r_s^2 ds \in du, \lambda \int_1^{1/t_2} r_s^2 ds \in dv) = \\
\int_\infty^{-\infty} \int_\infty^{-\infty} P^0 (t_1 r_{1/t_1} \in I_1, t_2 r_{1/t_2} \in I_2, \lambda \int_1^{1/t_1} r_s^2 ds \in du, \lambda \int_1^{1/t_2} r_s^2 ds \in dv) Q(\nu_u \in J_1, \nu_v \in J_2) = \\
\int_\infty^{-\infty} \int_\infty^{-\infty} P^0 (t_1 r_{1/t_1} \in I_1, t_2 r_{1/t_2} \in I_2, \lambda \int_1^{t_1} (sr_{1/s})^2 ds \in du, \lambda \int_1^{t_2} (sr_{1/s})^2 ds \in dv) Q(\nu_u \in J_1, \nu_v \in J_2) .
\]

Because (0.1) is true for \( (r_t) \) this is equal to

\[
\int_\infty^{-\infty} \int_\infty^{-\infty} P^0 (r_t \in I_1, r_t \in I_2, \lambda \int_1^{t_1} r_s^2 ds \in du, \lambda \int_1^{t_2} r_s^2 ds \in dv) Q(\nu_u \in J_1, \nu_v \in J_2) = \\
\int_\infty^{-\infty} \int_\infty^{-\infty} P^0 (r_t \in I_1, r_t \in I_2, \lambda \int_1^{t_1} r_s^2 ds \in du, \lambda \int_1^{t_2} r_s^2 ds \in dv) Q(\nu_u \in J_1, \nu_v \in J_2).
\]

Using Lemma 1 we get this equal to

\[
\int_\infty^{-\infty} \int_\infty^{-\infty} P^0 (r_t \in I_1, r_t \in I_2, \lambda \int_1^{t_1} r_s^2 ds \in du, \lambda \int_1^{t_2} r_s^2 ds \in dv) Q(\nu_u \in J_1, \nu_v \in J_2) = \\
P^0 \{r_t \in I_1, r_t \in I_2, \nu \in J_1, \nu \in J_2 \} = P^0 \{X_{t_1} \in (I_1, J_1), X_{t_2} \in (I_2, J_2) \}.
\]

\[\square\]

Remark: It would be interesting to know if the result still is true when \( \{0\} \) is not polar.

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